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Recall

DEF A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar function f i.e. $\vec{F} = \nabla f$. In this situation f is called a potential function for $\vec{F}$.

Theorem Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$.
Let f be a diff. function of two or three variables whose gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Defn If $\vec{F}$ is continuous vector field w/ domain D, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two path C_1, C_2 w/ same initial & terminal points.

- The line integrals of conservative vector fields are independent of path.

Theorem $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D.

Physical interpretation : Work done by conservative force field as it moves an object around a closed path is 0.

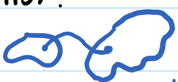
- So we know that if $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path.

Q Are there any non-conservative V.F. $\vec{F}$ s.t. $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path?

Ans No.

DEF D is said to be open, if for every point P in D is an interior point. (no boundary points).

D is said to be connected, if any two points in D can be joined by a path in D.



D : not connected.

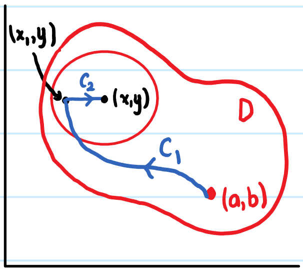
Thm Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D.

If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is conservative V.F on D i.e there exists a function f such that $\nabla f = \vec{F}$.

Proof Fix a point $(a,b) \in D$ and define $f(x,y) = \int_C \vec{F}\cdot d\vec{r}$, where C is any path from (a,b) to (x,y). (because of path independence).

- Now pick a disc centered at (x,y) contained in D (can do this since D is open)

Choose a point (x_1,y) in the disc with $x_1 < x$ and pick C to be any path from (a,b) to (x,y), call it C_1 followed by the horizontal line segment C_2 from (x_1,y) to (x,y)



Then, $f(x,y) = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

Note that first integral doesn't depend on x,

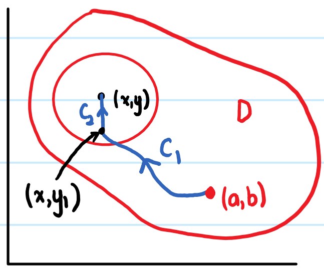
therefore $\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \vec{F}\cdot d\vec{r}$

If we write $\vec{F} = P\hat{i} + Q\hat{j}$, then we Know that $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} P\,dx + Q\,dy$

Note that on C_2 y doesn't change, so $dy = 0$. Then $C_2 = \langle t, y \rangle$ where $x_1 \le t \le x$.

Therefore, $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{C_2} P\,dx = \frac{\partial}{\partial x}\int_{x_1}^{x} P(t,y)\,dt \overset{FTC}{=} P(x,y)$.

Similarly, using a vertical line segment, we can show that:



$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y}\int_{y_1}^{y} Q(x,t)\,dt$$

$$= Q(x,y)$$

Therefore $\vec{F} = P\hat{i} + Q\hat{j}$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= \nabla f \qquad \blacksquare$$

Q Is it possible to determine whether or not a vector field $\vec{F}$ is conservative?

Assume $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative, where P and Q have continuous first-order partial derivatives.

Then there exists f such that $F = \nabla f$, i.e.

$P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

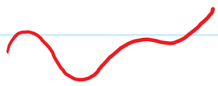
Therefore by Clairaut's Theorem, $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$

Theorem

If $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ is conservative, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$.

The converse is true only for a special type of regions.

Def Simple curve is a curve which doesn't intersect itself anywhere between the endpoints.



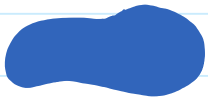
Simple, not closed

Not simple, not closed

closed, not simple

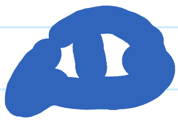
Closed and simple.

A simply connected region D is a connected region w/ no "holes".



Simply connected

not simply connected



not simply connected.

Thm let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field on an open simply connection region D.
Suppose that P and Q have continuous first-order derivatives and
$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D.

Then $\vec{F}$ is conservative.

Ex Determine whether or not $\vec{F}$ is a conservative vector field.

a) $\vec{F}(x,y) = e^x \cos y \,\hat{i} + e^x \sin y \,\hat{j}$

b) $\vec{F}(x,y) = (2x-3y)\hat{i} + (-3x+4y-8)\hat{j}$

a) $P(x,y) = e^x \cos y$, $Q(x,y) = e^x \sin y$

- $\frac{\partial P}{\partial y} = -e^x \sin y$, $\frac{\partial Q}{\partial x} = e^x \sin y$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ is not conservative.

b) $P(x,y) = (2x-3y)$, $Q(x,y) = -3x+4y-8$

$\frac{\partial P}{\partial y} = -3$, $\frac{\partial Q}{\partial x} = -3$

The domain of $\vec{F}$ is $\mathbb{R}^2$, which is open and simply conn, and so $\vec{F}$ is conservative.

So we know $\vec{F}(x,y) = (2x-3y)\hat{i} + (-3x+4y-8)\hat{j}$ is conservative, so how do we find the potential function f such that $\nabla f = F$.

We will use partial integration.

- $\nabla f = \vec{F} \Rightarrow \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (2x-3y)\hat{i} + (-3x+4y-8)\hat{j} \Rightarrow$

$\Rightarrow f_x(x,y) = 2x-3y \quad ; \quad f_y(x,y) = -3x+4y-8$

Integrating f_x wrt x, $f(x,y) = x^2 - 3xy + g(y)$

<u>Remark</u> The constant of integration is a constant with respect to x i.e. a function of y, which we call $g(y)$.

Differentiating $f(x,y)$ with respect to y gives $f_y(x,y) = -3x + g'(y)$.

Thus, $-3x+4y-8 = -3x+g'(y) \Rightarrow g'(y) = 4y-8$

Integrating wrt y, we have $g(y) = 2y^2 - 8y + K$.

So, $f(x,y) = x^2 - 3xy + 2y^2 - 8y + K$ is the desired potential function.

<u>Ex</u> Find a function f such that $\vec{F} = \nabla f$ where
$\vec{F}(x,y,z) = yz\,\hat{i} + xz\,\hat{j} + (xy+2z)\,\hat{k}$
and use your answer to evaluate $\int_C \vec{F}.d\vec{r}$ along
C is the line segment from
$(1,0,-2)$ to $(4,6,3)$.

<u>Soln</u> $\nabla f = \vec{F} \Rightarrow f_x(x,y,z) = yz$

$f_y(x,y,z) = xz \quad , \quad f_z(x,y,z) = (xy+2z)$

$f_x(x,y,z) = yz \Rightarrow f(x,y,z) = xyz + g(y,z)$

Then, $f_y(x,y,z) = xz + g_y(y,z)$

But $f_y(x,y,z) = xz \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$

Thus, $f(x,y,z) = xyz + h(z) \Rightarrow f_z(x,y,z) = xy + h'(z)$

But,

$$f_z(x,y,z) = xy + 2z \Rightarrow h'(z) = 2z \Rightarrow h(z) = z^2 + K$$

Hence, $f(x,y,z) = xyz + z^2 \quad (K=0)$

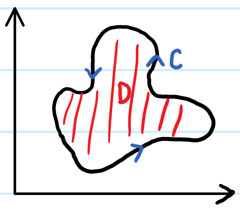
Then, $\int_C \vec{F}.d\vec{r} = \int_C \nabla f.d\vec{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$

Def Assume $\vec{F}$ is a conservative force field. Then the potential energy of an object at point (x,y,z) is a scalar function such that $\vec{F} = -\nabla P$.

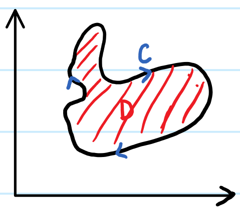
16.4 Green's Thm

We want to investigate the relationship between a line integral on a simple closed curve C and a double integral over the plane region D bounded by C.

Positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. If C is given by the vector function $\vec{r}(t)$, $a \le t \le b$, then the region D is always on the left as $\vec{r}(t)$ traverses C.



Positive orientation



Negative orientation

Green's Thm

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and D be the region bounded by C.

If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notation • $\oint_C P\,dx + Q\,dy$ or $\oint_C P\,dx + Q\,dy$ (indicating positive orientation of closed curve C)

• $\partial D \equiv$ positively oriented bdry curve of D

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{\partial D} P\,dx + Q\,dy$$

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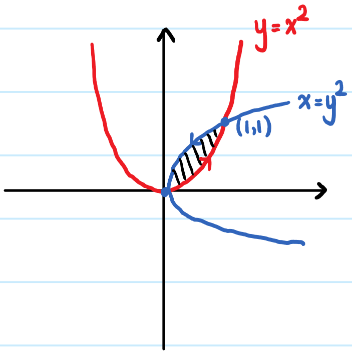
Think of it as an analogue of the FTC (Evaluation Theorem):

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives $\left(F', \frac{\partial Q}{\partial x} \text{ and } \frac{\partial P}{\partial y}\right)$ on LHS and RHS involves the values of original functions (F, Q and P) only on the boundary of the domain.

<u>Ex</u> Evaluate $\oint_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos y^2\right) dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

<u>Soln</u>



The region D bounded by C is the region $\{(x,y) \mid 0 \le y \le 1,\ y^2 \le x \le \sqrt{y}\}$

Then by Green's theorem,

$$\oint_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos y^2\right) dy = \iint_D \left[\frac{\partial}{\partial x}\left(2x + \cos y^2\right) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2-1)\,dx\,dy = \int_0^1 y^{1/2} - y^2\,dy = \left.\frac{2y^{3/2}}{3} - \frac{y^3}{3}\right]_0^1 = \frac{1}{3}.$$

- An application of the reverse direction of Green's Theorem is computing areas.

$A = \iint_D 1\,dA$, we wish to choose P and Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

<u>Several possibilities</u>

$P(x,y) = 0$ or $P(x,y) = -y$ or $P(x,y) = -\frac{1}{2}y$

$Q(x,y) = x$ $\quad Q(x,y) = 0$ $\quad Q(x,y) = \frac{1}{2}x$

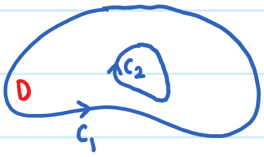
Then, $A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$

Ex Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln The parametric equation for ellipse is $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

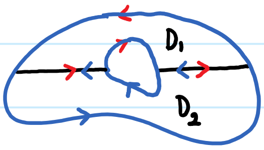
$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

- Green's theorem can be applied to region w/ holes (i.e regions that are not simple connected.



The boundary C of region D consists of two simple closed curves C_1 and C_2.

Assume C is oriented so that the region D is always on the left as we travel along C.
Divide region D into two pieces D_1 and D_2 as shown and apply Green's Theorem to both D_1 & D_2.



$$\iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA = \iint_{D_1} \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA$$

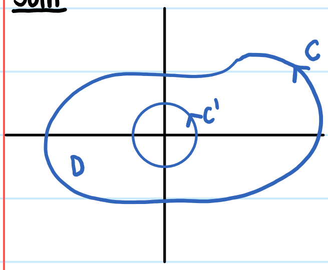
$$= \iint_{\partial D_1} P\,dx + Q\,dy + \iint_{\partial D_2} P\,dx + Q\,dy$$

As the line integrals along the common boundary are in opposite direction, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy.$$

Ex If $\vec{F}(x,y) = \frac{(-y\hat{i} + x\hat{j})}{(x^2+y^2)}$, show that $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Soln



C = positively oriented simple closed curve that encloses the origin.
No idea how to compute integral directly.

Instead pick a counterclockwise oriented circle C' w/ center $(0,0)$ and radius a so that C' lies inside C.

Let D be the region enclosed by C and C'.
Then the positively oriented boundary of D is $C \cup -C'$ and Green's Thm gives us

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\, dA$$

$$= 0$$

Then $\displaystyle\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$$

Now C' : $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$, $0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \left\langle \frac{-a\sin t}{a^2\cos^2 t + a^2\sin^2 t}, \frac{a\cos t}{a^2\cos^2 t + a^2\sin^2 t} \right\rangle \cdot \langle -a\sin t, a\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \frac{a^2\sin^2 t + a^2\cos^2 t}{a^2}\, dt = \int_0^{2\pi} dt = 2\pi \quad \blacksquare$$